Study of the equilibria of high-dimensional equations describing ecosystems' dynamics

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Certain ecosystems (plants, bacteria...) are very high-dimensional: they are made of a huge number N of species which coexist in the same spatial region, interacting with each others. The interactions between all the possible pairs (or groups) of species are in general difficult to measure and to infer: in a statistical physics approach, one can model the interaction couplings with random variables, introducing an *ensemble* of ecosystems to be studied statistically [1]. In this setting, the problem can be treated with techniques of the statistical physics of disordered systems.
A very well-studied model to describe species-rich, well-mixed ecosystems are the generalized Lotka-Volterra equations. We let $i = 1, \cdots, N$ label each possible species, and $n_i(t)$ denote the number of individuals of species i that is present at time t in a given realization of the ecosystems. This number evolves as:

$$\frac{dn_i(t)}{dt} = n_i(t) \left[\kappa_i - n_i(t) - \sum_j \alpha_{ij} n_j(t) \right], \tag{1}$$

where κ_i, a_{ij} are random variables with Gaussian statistics, and $N \gg 1$ (see [2] for a review). For certain values of parameters (in particular, when the variance controlling the fluctuations of the a_{ij} is large), these dynamical equations admit an extremely large number of equilibrium configurations, i.e. of high-dimensional vectors $\mathbf{n}^* = (n_1^*, \cdots, n_N^*)$ such that

$$\frac{dn_i(t)}{dt}\Big|_{\mathbf{n}^*} = n_i^* \left[\kappa_i - n_i^* - \sum_j \alpha_{ij} n_j^* \right] = 0. \tag{2}$$

In order to understand the dynamics of the system at large time scales, an important question to address is how many are these equilibria, and what are their properties (in particular, how are equilibria distributed in terms of their diversity, i.e. the fraction of species for which $n_i^* > 0$).
This study of equilibria can be addressed using a combination of tools of the statistical physics of disordered systems, such as the replica formalism and random matrix theory. In particular, the typical number of equilibria at a given diversity can be computed in the limit of large N, by solving a family of coupled self-consistent equations. These equations have been derived in [3, 4], and solved in the particular limit of uncorrelated couplings a_{ij}. The goal of this project is to extend the solution to a whole regime of parameters controlling the statistical properties of the interactions a_{ij} between the species. This involves some analytical work (understand the main steps in the derivation of the equations) as well as some numerical work (obtain numerically the solutions for different values of parameters, deriving a phase diagram).

References

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